18 Suppose V is finite-dimensional, with $\dim V = n \geq 1$. Prove that there exist one-dimensional subspaces $V_1, \dots, V_n$ of V such that

$$V = V_1 \oplus \cdots \oplus V_n.$$

Let $(\gamma_1, \dots, \gamma_n)$ be a basis for V.

Let $U_i = \operatorname{span}(\gamma_i)$ for $i = 1, 2, \dots, n$

Since $\gamma_i \neq 0$ for $i = 1, 2, \dots, n$

$\dim(\gamma_i) = 1$.

Claim 1: $V \subseteq U_1 + U_2 + \dots + U_n$

Let $\gamma \in V$. Then since $\gamma_1, \dots, \gamma_n$ is a basis of V, there exist $a_1, \dots, a_n \in \mathbb{F}$ such that

$$\gamma = a_1 \gamma_1 + \dots + a_n \gamma_n \quad \text{———} \; (*)$$

Then $a_i \gamma_i \in U_i$, $i = 1, 2, \dots, n$

Then $\gamma = U_1 + U_2 + \dots + U_n$

Thus $V \subseteq U_1 + \dots + U_n$ ——— (1)

We know that $U_1 + \dots + U_n$ is subspace of V. ~~Th~~ ~~i.e.~~ $U_1 + U_2 + \dots + U_n \subseteq V$ ——— (2)

By (1) and (2)

$$V = U_1 + \dots + U_n \quad \text{———} \; (**)$$

Claim 2: $U_i \cap U_j = \{0\}$ $i, j = 1, 2, \dots, n$ and $i \neq j$

Assume the contrary $0 \neq u \in U_i \cap U_j$

$u \in V_i \Rightarrow u = a_i \gamma_i$ forsome $a_i \in \mathbb{F}$

$u \in V_j \Rightarrow u = a_j \gamma_j$ forsome $a_j \in \mathbb{F}$

~~Since~~ Then $u = a_i \gamma_i = a_j \gamma_j$

$$a_i \gamma_i - a_j \gamma_j = 0$$

Since, γ_i, γ_j are linearly independent

$$\cancel{a_i \neq 0} \quad a_i = a_j = 0$$

Therefore, $\cancel{u = a_i \gamma_i = a_j \gamma_j}$

$$u = 0$$

Thus $V_i \cap V_j = \{0\}$ for $i, j = 1, 2, \dots, n$ $i \neq j$ ***

~~Thus $V_1, \dots, V_n$ are pa~~

Therefore $V = V_1 \oplus \dots \oplus V_n$

By ** and ***

19 Explain why you might guess, motivated by analogy with the formula for the number of elements in the union of three finite sets, that if V_1, V_2, V_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim(V_1 + V_2 + V_3)& \\ &= \dim V_1 + \dim V_2 + \dim V_3 \\ &\quad - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) \\ &\quad + \dim(V_1 \cap V_2 \cap V_3).\end{aligned}$$

Then either prove the formula above or give a counterexample.

Counterexample: Let $U_1 = \text{span}\{(1,0)\}$
$U_2 = \text{span}\{(0,1)\}$
$U_3 = \text{span}\{(1,1)\}$

Note that,

$$\dim(U_1) = \dim(U_2) = \dim(U_3) = 1$$

Now observe that.

$$\emptyset = U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = U_1 \cap U_2 \cap U_3.$$

$$0 = \dim(U_1 \cap U_2) = \dim(U_1 \cap U_3) = \dim(U_2 \cap U_3) = d(U_1 \cap U_2 \cap U_3)$$

Furthur,

<u>Claim: $U_1 + U_2 + U_3 = \mathbb{R}^2$</u>

Let $(x_1, x_2) \in \mathbb{R}^2$. Then

$(x_1, x_2) = (x_1 - 1, 0) + (0, x_2 - 1) + (1,1) \in U_1 + U_2 + U_3$.

Because, $(x_1 - 1, 0) \in U_1$, $(0, x_2 - 1) \in U_2$, $(1,1) \in U_3$

Thus, $\mathbb{R}^2 \subseteq U_1 + U_2 + U_3$.

It is trivial that, $\mathbb{R}^2 \supseteq U_1 + U_2 + U_3$. Therefore, $\mathbb{R}^2 = (U_1 + U_2 + U_3)$.

$$\dim(U_1 + U_2 + U_3) = 2 \quad \text{—— } (*)$$

According to given formula,

$$\begin{aligned}\dim(U_1 + U_2 + U_3) = &\dim(U_1) + \dim(U_2) + \dim(U_3) \\ &- \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ &+ \dim(U_1 \cap U_2 \cap U_3)\end{aligned}$$

$$= (1 + 1 + 1) - (0 + 0 + 0) + 0$$

$$= 3 \quad \text{—— } (**)$$

$(*)$ and $(**)$, give a contradiction.

20 Prove that if V_1, V_2, and V_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}
&\dim(V_1 + V_2 + V_3) \\
&\quad = \dim V_1 + \dim V_2 + \dim V_3 \\
&\qquad - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3} \\
&\qquad - \frac{\dim((V_1+V_2)\cap V_3) + \dim((V_1+V_3)\cap V_2) + \dim((V_2+V_3)\cap V_1)}{3}.
\end{aligned}$$

The formula above may seem strange because the right side does not look like an integer.

By 2.43 we can obtain following

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$$\begin{aligned}
\dim(V_1+V_2+V_3) &= \dim(V_1+V_2) + \dim(V_3) - \dim((V_1+V_2)\cap V_3) \\
&= \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2) \\
&\quad + \dim(V_3) - \dim((V_1+V_2)\cap V_3) \qquad \text{I}
\end{aligned}$$

Similarly

$$\begin{aligned}
\dim(V_1+V_2+V_3) &= \dim(V_1) + \dim(V_2) + \dim(V_3) \\
&\quad - \dim(V_1 \cap V_3) - \dim((V_1+V_3)\cap V_2) \qquad 2
\end{aligned}$$

$$\dim(V_1+V_2+V_3) = \dim(V_1) + \dim(V_2) + \dim(V_3) - \dim(V_2 \cap V_3) - \dim((V_2+V_3)\cap V_1) \quad \text{---} \ \overset{3}{\cancel{4}}$$

(1) + (2) + (3)

$$3\dim(V_1+V_2+V_3) = 3\dim(V_1) + 3\dim(V_2) + 3\dim(V_3) - \dim(V_1\cap V_2) - \dim(V_2 \cap V_3) - \dim(V_1 \cap V_3) - \dim((V_1+V_2)\cap V_3) - \dim((V_1+V_3)\cap V_2) - \dim((V_2+V_3)\cap V_1)$$

$$\dim(V_1+V_2+V_3) = \dim(V_1) + \dim(V_2) + \dim(V_3) - \frac{1}{3}\left[\dim(V_1\cap V_2) + \dim(V_2\cap V_3) + \dim(V_1\cap V_3)\right] - \frac{1}{3}\left[\dim((V_1+V_2)\cap V_3) + \dim((V_1+V_3)\cap V_2) + \dim((V_2+V_3)\cap V_1)\right]$$